

Core equivalence in economies with satiation

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Abstract

It has long been known that in economies with satiation, the set of competitive equilibria does not coincide with the limiting core of an economy. Competitive equilibria may fail to exist because satiated agents may choose their optimal consumption bundles in the interiors of their budget sets, creating a total budget excess. On the other hand, the core of the economy grows very large. Satiated agents are not interested in entering any coalition; non-satiated agents lack the resources for blocking.

In dividend equilibria, introduced independently by a number of authors, the budget excess is allowed to be divided among consumers as dividends and equilibrium existence is restored. In this paper we introduce a new notion of blocking, which leads to core - dividend equilibrium equivalence. It is shown that under a condition of strict monotonicity, both revised and traditional notions of blocking lead to the same fuzzy core.

Key words: satiation, fixed prices, dividend equilibrium, fuzzy core

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1 Introduction

Consider an exchange economy where the preferences of some agents are satiated over some goods (for example, over the number of consumed hours of leisure). In this case, competitive equilibria may fail to exist, since no matter what the prices are, agents may choose their optimal consumption bundles in the interiors of their budget sets, which results in the creation of a total budget excess and a violation of Walras Law.

In dividend equilibria, introduced in Drèze and Müller (1980), Makarov (1981), Aumann and Drèze (1986), and Mas-Colell (1992), the budget excess is allowed to be divided among consumers as dividends, and equilibrium existence is restored. Each consumer's budget is then the sum of his dividend and the value of his initial endowments. If a budget surplus is equally divided among all non-satiated agents, we refer to such a dividend equilibrium as uniform. Since for locally non-satiated preferences a budget excess disappears and a dividend equilibrium is a usual competitive equilibrium, it can be treated as an extension of Walrasian equilibrium.

To illustrate this situation, consider the following simple example of a pure exchange economy with three consumers and one consumption good. The consumption set is the set of non-negative numbers for each consumer. All consumers are endowed with 2 units of the consumption good. Preferences of consumers are represented by utility functions $u_1(x) = -x$ and $u_2(x) = u_3(x) = x$. There is no Walrasian equilibrium in this economy, a continuum of dividend equilibria $x^1 = 0$, $x^2 = 2 + \lambda$, $x^3 = 4 - \lambda$, for $\lambda \in [0, 2]$, and a single uniform dividend equilibrium $x^1 = 0$, $x^2 = x^3 = 3$.

The analysis of satiated preferences is relevant whenever the choice set of a decision maker is naturally compact. The major example of an economy with satiation is a fixed price economy, where all trade is restricted to take place at exogenously given fixed (at least, in the short-run) prices p . In such an economy, satiated preferences over the set of all attainable at given p consumption bundles are induced by initially non-satiated preferences of agents. Another example of a satiated economy is a model of incomplete asset markets where investors maximize expected utility and the total return to individual assets may be negative with positive probability, see Nielsen (1994).

Note that there is a great deal of literature devoted to the related problem of distributing a bundle of commodities among a group of agents who are collectively entitled to them and whose preferences are single-peaked (see, for

example, Thomson and Zhou (1993)). Incorporating initial endowments into the one-dimensional case of this problem led Klaus, Peters, and Storcken (1998) to the notion of the uniform reallocation rule that yields the same allocations of goods as uniform dividend equilibrium does.

The (strong) core of an economy is the set of all feasible allocations which no coalition can block by reallocating its resources among its members in a way that all of them become strictly better off. In traditional models of an economy without satiation or price rigidities, there is a close relationship between competitive equilibria and the core. Namely, under standard assumptions, equilibrium allocations belong to the core and, conversely, with an increasing number of agents, the core converges to the set of competitive equilibria (see Debreu and Scarf (1963), Anderson (1992)). The aim of the present paper is to extend this result to the case of an economy with satiation. For this purpose, we introduce a new notion of blocking, which leads to core - dividend equilibrium equivalence.

It should be mentioned that there is another notion of blocking often used in the defining the core of an economy, which requires that at least one member of a blocking coalition be made better off without any agent worse off. With monotonicity of preferences, both notions are equivalent; without monotonicity they need not be so. As was pointed out in Aumann and Drèze (1986), the limiting weak core coincides with the set of competitive equilibria, and, since for markets with satiation there are typically no competitive equilibria, is then empty. On the other hand, if non-satiation of preferences fails, the strong core of an economy grows very large. Satiated agents are not interested in entering any coalition; non-satiated agents lack the resources for blocking. The next section contains two examples which illustrate this situation. In Section 3, the concept of blocking is revised in an appropriate way by giving more blocking power to non-satiated agents. In Section 4, we introduce, following Aubin (1979), the concept of the fuzzy core, which can be considered as a limiting core of an economy. It is shown that if preferences of agents are strictly monotone then the revised blocking (which we call rejection) results in the same set of the fuzzy core allocations as the usual blocking does. Section 5 deals with the formal definition of a dividend equilibrium. The main result of the paper (the equivalence theorem) is stated and proved in the last section.

2 Motivating examples

In an economy with satiation some goods that do not improve the utility of any member of a certain coalition can be transferred to another one. That coalition improves in this way its situation not by means of an inside trading process, but with the aid of the subsidy received. It implies an extension of the resources that a blocking coalition is permitted to use. For example, it is natural to think that a coalition blocks some allocation if it can redistribute the part of *this* allocation (not its initial resources) to which it is entitled among its members in a Pareto improving way.

To illustrate the point, consider an economy with two goods x_A and x_B (apples and bananas), and three consumers; the first of them is fruit averse, $u_1(x_A, x_B) = -(x_A + x_B)$, the second agent is apples loving and bananas indifferent, $u_2(x_A, x_B) = x_A$, and the third one is bananas loving and apples neutral, $u_3(x_A, x_B) = x_B$. The economy is endowed with one apple and one banana that are possessed by the fruit allergic consumer. All other consumers have initial endowments equal to zero. Short sales of commodities are not allowed.

Suppose that the first consumer has handed the apple to the banana-lover and, vice versa, the banana to the apple-lover. The resulting allocation of goods $x^1 = (0, 0)$, $x^2 = (0, 1)$, $x^3 = (1, 0)$ belongs to the core. Indeed, the satiated agent will withdraw from participating in any coalition; the other two agents can gain only utility zero from reallocating their zero initial endowments. But non-satiated consumers actually can improve this situation, first, by accepting the first agent's offer, and second, after that, by trading between themselves. Mathematically speaking, there exist vectors of commodity bundles $y^2 = (1, 0)$, and $y^3 = (0, 1)$, which are strictly better than x^2 and x^3 for agents 2 and 3 respectively, such that

$$y^2 + y^3 = x^2 + x^3.$$

This is precisely a Pareto-improving reallocation of $x = (x^1, x^2, x^3)$ according to the definition of Pareto optimality given in the case of possible satiation of agents' preferences as in Marakulin (1988) and (1990). Note that the weak core in this example is empty starting already from the second replica of an economy.

However, to provide a core-equilibrium equivalence result, a further extension of a coalition's blocking power is required. Consider one more ex-

ample with one satiated and two identical non-satiated consumers. There are two goods (x_1 and x_2) in the economy, consumption sets are positive orthants and initial endowments are $w^1 = (2, 0)$, $w^2 = w^3 = (0, 1)$. The first agent has a bliss point at $(0, 2)$, his preferences are given by $u_1(x_1, x_2) = 4 - (x_1)^2 - (x_2 - 2)^2$; the other two agents' preferences are given by Leontief utility functions: $u_2(x_1, x_2) = u_3(x_1, x_2) = \min\{x_1, x_2\}$.

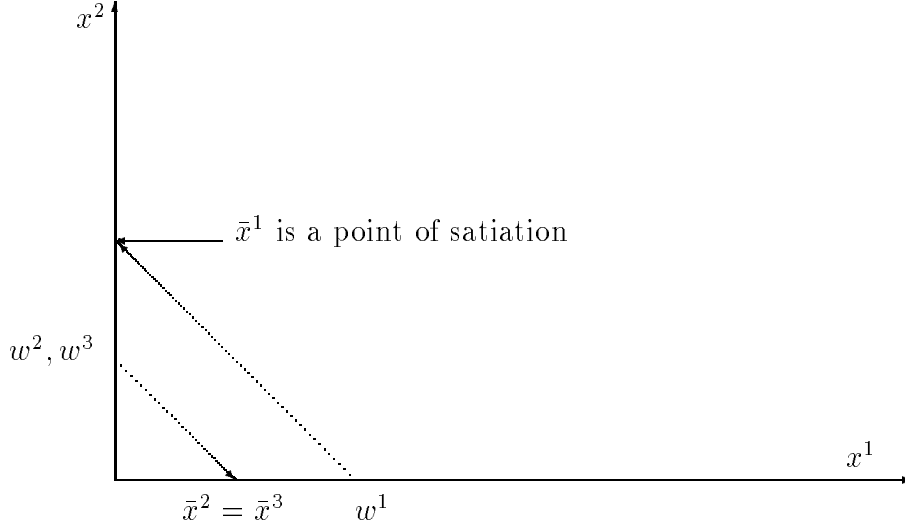


Figure 1: Example.

Consider the allocation of goods $\bar{x}^1 = (0, 2)$, $\bar{x}^2 = \bar{x}^3 = (1, 0)$. It belongs to the core, because the first agent is at his satiation point, while the others can obtain only zero utility from redistributing their initial holdings. Moreover, this allocation belongs to the core of any replicated economy; the satiated consumers are not interested in entering any coalition; the non-satiated lack the resources for blocking.

On the other hand, this allocation can not be sustained by any equilibrium price system for the following reason. Since a Walrasian budget set (with or without a dividend) is convex and contains the agent's initial endowments w^i , it should also contain any convex combination of w^i and his preference optimum on this set (a Walrasian demand). Therefore, neither \bar{x}^2 nor \bar{x}^3 is a preference optimum subject to any budget constraint, whatever the prices are.

On the third hand, imagine the coalition of agents 2 and 3 discussing over

the offer (\bar{x}^2, \bar{x}^3) from the complimentary coalition (agent 1 in the present instance). For example, agent 2 may be conditioned to say to his fellow participant: 'Let you accept the offer of agent 1 and get \bar{x}^3 , whereas I reject it and stay with w^2 . Then we will get $w^2 + \bar{x}^3 = (1, 1)$ and going halves become better off.' In other words, there exist consumption bundles $y^2 = y^3 = (1/2, 1/2)$, such that

$$u_2(y^2) > u_2(\bar{x}^2), \quad u_3(y^3) > u_3(\bar{x}^3),$$

$$y^2 + y^3 = w^2 + \bar{x}^3.$$

If such reasoning takes place, the allocation $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$ will not be realized, and this is the sense in which we will be speaking about blocking of undesirable allocations throughout the paper.

3 The model

Consider an exchange economy

$$\mathcal{E} = \langle N, L, \{X_i, \mathcal{P}_i, w^i\}_{i \in N} \rangle.$$

Here $N = \{1, \dots, n\}$ is the set of consumers; L is the (positive integer) number of goods. For each agent $i \in N$ his consumption set X_i is a convex subset of \mathbb{R}^L , and his vector of initial endowments w^i belongs to the interior of X_i . A correspondence $\mathcal{P}_i : X_i \rightarrow 2^{X_i}$ gives the preferences of agent i , that is for $x^i \in X_i$, $\mathcal{P}_i(x^i)$ denotes the consumption bundles that are (strictly) preferred to x^i . The preferences of agents are assumed to be convex, continuous and irreflexive, that is for every $i \in N$ and for every $x^i \in X_i$ the set $\mathcal{P}_i(x^i)$ is convex, open in X_i and does not contain x^i . We allow satiated preferences, that is emptiness of the set $\mathcal{P}_i(x^i)$ for some $x^i \in X_i$, $i \in N$.

As usual, we say that an allocation $x = (x^i)_{i \in N}$ in $X = \prod_{i \in N} X_i$ is *feasible* if

$$\sum_{i \in N} x^i = \sum_{i \in N} w^i.$$

The reasoning of the previous section motivates the following definition.

Definition 1 *A coalition $S \subseteq N$ rejects an allocation $\bar{x} \in X$, if there exist subcoalitions $S_1 \subseteq N$, $S_2 \subseteq N$ and consumption bundles $y^i \in X_i$, $i \in S$, such that*

- (i) $S_1 \cup S_2 = S, \quad S_1 \cap S_2 = \emptyset,$
- (ii) $\sum_{i \in S} y^i = \sum_{i \in S_1} w^i + \sum_{i \in S_2} \bar{x}^i,$
- (iii) $y^i \in \mathcal{P}_i(\bar{x}^i), \quad i \in S.$

Note that the introduced notion of rejection is stronger than the traditional notion of blocking that can be obtained by putting $S_2 = \emptyset, S = S_1$. Since the resulting "mixed" vector of coalition's resources $((w^i)_{i \in S_1}, (\bar{x}^i)_{i \in S_2})$ may not be attainable in an economy, "rejection" seems to be a more appropriate term here than "blocking" or, especially, "improving upon". A coalition may not really be able to "improve upon" the allocation, but it can still reject it.

The *rejective core* of an economy \mathcal{E} is the set of all feasible allocations which can not be rejected by any coalition.

4 The fuzzy core

An element $\xi = (\xi_i)_{i \in N}$ of the set $[0, 1]^N \setminus \{0\}$ is called a *fuzzy coalition*. Its i th component ξ_i was interpreted by Aubin (1979) as *the rate of participation* of agent i in this fuzzy coalition. Specifically, unlike the usual case, where ξ_i is either 1 or 0, that is an agent i either participates entirely in the coalition

$$\text{supp } \xi = \{i \in N \mid \xi_i > 0\}$$

or does not participate in it at all, members of a fuzzy coalition may contribute to it any share of its initial endowments.

Definition 2 A fuzzy coalition ξ blocks an allocation $\bar{x} \in X$, if there exist $y^i \in X_i, i \in \text{supp } \xi$, such that

- (i) $\sum_{i \in N} \xi_i y^i = \sum_{i \in N} \xi_i w^i,$
- (ii) $y^i \in \mathcal{P}_i(\bar{x}^i), \quad i \in \text{supp } \xi.$

Another interpretation of fuzzy coalitions is as follows. Suppose for a moment that N is the set of agents' *types* and that a continuum set of agents of each type is associated with the set $[0, 1]$. In this "large" atomless economy ξ_i represents the measure of the set of all agents of type i participating in a certain coalition, whose average aggregate resources are summed up to

$\sum_{i \in N} \xi_i w^i$. Thus one may consider a *fuzzy core* (that is the set of all feasible allocations that can not be blocked by any fuzzy coalition) as a "limit" core of the initial economy \mathcal{E} .¹ No wonder that the Edgeworth's conjecture is true for the fuzzy core: it is easy to show that under the standard assumptions of the Debreu-Scarf theorem (convex, continuous, locally non-satiated preferences; initial endowments are in the interiors of consumption sets) the fuzzy core coincides with the set of competitive equilibria. In the present paper we prove a similar result for the set of dividend equilibria without assuming local non-satiation.

Combining the two notions given by Definitions 1 and 2 one obtains the following definition of fuzzy rejection.

Definition 3 *A fuzzy coalition $\xi = (\xi_i)_{i \in N}$ rejects an allocation $\bar{x} \in X$, if there exist $y^i \in X_i$, $t_i \in \mathbb{R}_+$, $q_i \in \mathbb{R}_+$, $i \in \text{supp } \xi$, such that*

- (i) $t_i + q_i = \xi_i$, $i \in N$,
- (ii) $\sum_{i \in N} \xi_i y^i = \sum_{i \in N} t_i w^i + \sum_{i \in N} q_i \bar{x}^i$,
- (iii) $y^i \in \mathcal{P}_i(\bar{x}^i)$, $i \in \text{supp } \xi$.

Here (i) is the analogue of (i) in Definition 1. A fuzzy coalition ξ is splitted into two parts, all agents from one part are supposed to accept \bar{x} ; the other members of ξ are supposed to reject \bar{x} and stay with their initial endowments.

Definition 4 *The fuzzy rejective core of an economy \mathcal{E} is the set of all feasible allocations $x \in X$, which are not rejected by any fuzzy coalition.*

If preferences of agents are strictly monotone, then the fuzzy rejective core coincides with the fuzzy core of an economy.

Proposition 1 *Assume that the consumption sets of agents are positive orthants: $X_i = \mathbb{R}_+^L$, $i \in N$, and for every $x^i, y^i \in X_i$ such that $x^i \neq y^i$, $y^i \in \{x^i\} + \mathbb{R}_+^L$ implies $y^i \in \mathcal{P}_i(x^i)$. Then the fuzzy rejective core of an economy \mathcal{E} coincides with the fuzzy core of \mathcal{E} .*

¹It can be shown that the fuzzy core is the set of all core elements in a "large" economy, that satisfy the equal-treatment property, see Hüsseinov (1994).

Proof. Let $\bar{x} \in X$ be some arbitrary feasible allocation. It is sufficient to show that fuzzy rejection of \bar{x} implies its fuzzy blocking. Suppose that there exist a fuzzy coalition ξ and consumption bundles $y^i \in \mathcal{P}_i(\bar{x}^i)$, $i \in \text{supp } \xi$, such that

$$\sum_{i \in N} \xi_i y^i = \sum_{i \in N} t_i w^i + \sum_{i \in N} q_i \bar{x}^i \quad (1)$$

for some nonnegative $t_i, q_i, i \in N$ such that $\xi_i = t_i + q_i$. Consider $q_{\max} = \max_{i \in N} q_i$. Without loss of generality $q_{\max} \neq 0$. Rewrite (1) as follows

$$\sum_{i \in N} t_i y^i + \sum_{i \in N} q_{\max} \left(\frac{q_i}{q_{\max}} y^i + \left(1 - \frac{q_i}{q_{\max}}\right) \bar{x}^i \right) = \sum_{i \in N} t_i w^i + \sum_{i \in N} q_{\max} \bar{x}^i. \quad (2)$$

By feasibility of \bar{x} the right-hand part of (2) is equal to

$$\sum_{i \in N} (t_i + q_{\max}) w^i.$$

Denote by y^i the convex combination of vectors y^i and \bar{x}^i in the left-hand side of (2) :

$$y^i = \frac{q_i}{q_{\max}} y^i + \left(1 - \frac{q_i}{q_{\max}}\right) \bar{x}^i.$$

By openness of $\mathcal{P}_i(\bar{x}^i)$, for every i such that $q_i \neq 0$ we can find $\alpha > 0$ and commodity l , such that $y^i - \alpha e_l \in \mathcal{P}_i(\bar{x}^i)$, where e_l denotes the l -th unit vector in \mathbb{R}^L . Then every y^i such that $q_i \neq q_{\max}$ can be represented as a convex combination of two vectors that are strictly preferable to \bar{x}^i :

$$y^i = \frac{q_i}{q_{\max}} (y^i - \alpha e_l) + \left(1 - \frac{q_i}{q_{\max}}\right) (\bar{x}^i + \beta e_l),$$

where $\beta = \alpha q_i / (q_{\max} - q_i)$. Therefore $y^i \in \mathcal{P}_i(\bar{x}^i)$. Define

$$z^i = \frac{t_i}{t_i + q_{\max}} y^i + \frac{q_{\max}}{t_i + q_{\max}} y^i, \quad i \in N,$$

$$\xi'_i = \frac{t_i + q_{\max}}{\max_{j \in N} (t_j + q_{\max})}, \quad i \in N.$$

Then (2) can be rearranged as

$$\sum_{i \in N} \xi'_i z^i = \sum_{i \in N} \xi'_i w^i.$$

Notice, that every z^i is either equal to \bar{x}^i if $t_i + q_i = 0$ or strictly better than \bar{x}^i otherwise. Without loss of generality $z^1 \in \mathcal{P}_i(\bar{x}^1)$, $\xi'_1 > 0$. Again, we can find $l \in L$ and $\alpha > 0$ such that $z'^1 = z^1 - \alpha e_l \in \mathcal{P}_i(\bar{x}^1)$. Distribute amount $\alpha \xi'_1$ of good l among the rest of the fuzzy coalition ξ' :

$$z'^i = z^i + \frac{\alpha \xi'_1 e_l}{\xi'_i(|\text{supp } \xi'| - 1)}, \quad i \in \text{supp } \xi' \setminus \{1\}.$$

Now we have

$$\sum_{i \in N} \xi'_i z'^i = \sum_{i \in N} \xi'_i w^i,$$

where $z'^i \in \mathcal{P}_i(\bar{x}^i)$ for every $i \in \text{supp } \xi'$. The proposition is proved.

□

5 Dividend equilibria

The notion of dividend equilibrium allows us to distribute the surplus created by the satiated agents among the non-satiated agents. It is done by relaxing each consumer's budget constraint by some slack variable that can be interpreted as an extra amount of income to spend. The budget of an agent is then the sum of the value of his initial endowments and this additional income term (a dividend).

Definition 5 *A feasible allocation $\bar{x} = (\bar{x}^i)_{i \in N} \in X$ is a dividend equilibrium if there exist $d = (d_1, \dots, d_n) \in \mathbb{R}_+^N$ and $p \in \mathbb{R}^L$ such that for all $i \in N$, \bar{x}^i maximizes i th preferences over his dividend budget set*

$$B_i(p, w^i, d_i) = \{x \in X_i \mid px \leq pw^i + d_i\},$$

in other words the following conditions hold:

(i) *attainability* —

$$\bar{x}^i \in B_i(p, w^i, d_i), \quad i \in N,$$

(ii) *individual rationality* —

$$\mathcal{P}_i(\bar{x}^i) \cap B_i(p, w^i, d_i) = \emptyset, \quad i \in N.$$

When $d_i = d$ for all $i \in N$, a dividend equilibrium is uniform. If $d = 0$, then the above is the usual definition of a Walrasian equilibrium.

The first use of the concept of dividend equilibrium can be traced as far back as 1959. Debreu in his "Theory of Value" used *equilibria relative to the price system* that can be viewed as dividend equilibria with possibly negative dividends in order to prove the first and the second welfare theorems.²

Drèze and Müller (1980) introduced dividend equilibria in a fixed-price model. Following Debreu (1959), they showed that these equilibria are Pareto-optimal (efficient) and, conversely, each Pareto-optimal allocation is generated by some system of dividends and prices.

Makarov (1981) considered dividend equilibria in economies with production, calling them "equilibria with transferable values", and proved their existence under rather weak assumptions (without ordered preferences, free disposal, any special form of income distribution functions, etc.). To guarantee the existence of dividend equilibria we need, in terms of this paper, all consumption sets to be closed and all the correspondences \mathcal{P}_i to have an open graph. We shall prove our equivalence result under slightly weaker conditions, assuming only openness of the images $\mathcal{P}_i(x^i)$ for every $x^i \in X_i$, $i \in N$.

The paper of Aumann and Drèze (1986) revealed the relationship between uniform dividend equilibria and Shapley value allocations in large economies.

Mas-Colell (1992) obtained results similar to those of Makarov and proved that there always exists a uniform dividend equilibrium which is a strong Pareto optimum under conditions weaker than those of Drèze and Müller (1980).

Kajii (1996) showed that in economies with satiation competitive equilibria do exist if one of the commodities is paper money (the good all consumers' utilities are independent of). Satiation points become "satiation lines" then, and moving along such a line a consumer can arrive at the boundary of his budget set. This approach makes it possible to consider dividend equilibria as Walrasian equilibria in an extended economy.

Let us return to the example of the introduction and make every household possess one unit of paper money. Consider the economy where consumption sets are positive orthants and utility functions are defined as above. Then there exist a unique competitive equilibrium such that the relative price of

²Thus, one can observe that under the standard assumptions, the set of dividend equilibria coincides with the set of individually rational Pareto optima.

the first (non-money) good is equal to 1, consumer 1 collects all the money and consumers 2 and 3 receive one unit of consumption good from consumer 1 each. Here though the paper money is useless, the satiated agent is ready to accept it until the budget constraint is met. Note that the equilibrium allocation of the first good is the same as in the uniform dividend equilibrium. This is caused by the initial uniform distribution of money, but as far as the existence of equilibria is concerned, the distribution of initial endowments of money is irrelevant. Though in many cases (as in the case of a fixed price economy) there is no natural way to pour paper money into an economy, this approach provides us with a good conventional view on a mechanism of budget excess redistribution.

Finally, generic finiteness of the set of uniform dividend equilibria is the easy corollary of the generic finiteness result for the set of uniform dividend equilibria with non-standard prices established in Konovalov and Marakulin (1997).

To analyse the situation where non-satiation of preferences fails and markets are incomplete, Polemarchakis and Siconolfi (1993) suggested the concept of a weak competitive equilibrium. In such an equilibrium budget constraints are defined in the form of equality so that the consumers are forced to spend all their income and Walras Law is restored by construction. Since those equilibria are not efficient, whereas we are looking for the properties which are stronger than mere efficiency, we leave them out of consideration here.

6 An equivalence result

Denote the fuzzy rejective core by $C_{fr}(\mathcal{E})$ and the set of dividend equilibria by $W_d(\mathcal{E})$. Now we are ready to formulate the main result of the paper.

Theorem 1 *Assume that for every $i \in N$, X_i is convex, w^i belongs to the interior of X_i , and for each $x^i \in X_i$ the set $\mathcal{P}_i(x^i)$ is convex, open in X_i and does not contain x^i . Then the fuzzy rejective core of the economy \mathcal{E} coincides with the set of dividend equilibria:*

$$C_{fr}(\mathcal{E}) = W_d(\mathcal{E}).$$

Proof of theorem. It is easy to show that every dividend equilibrium belongs to the fuzzy rejective core. Suppose otherwise. Let $\bar{x} \in X$ be a

dividend equilibrium, such that for some fuzzy coalition $(\xi_i)_{i \in N} = (t_i)_{i \in N} + (q_i)_{i \in N}$ and for some consumption bundles $y^i \in \mathcal{P}_i(\bar{x}^i)$, $i \in \text{supp } \xi$,

$$\sum_{i \in N} \xi_i y^i = \sum_{i \in N} t_i w^i + \sum_{i \in N} q_i \bar{x}^i. \quad (3)$$

Given the equilibrium prices p and system of dividends d , individual rationality of \bar{x} implies $y^i \notin B_i(p, w^i, d_i)$, $i \in \text{supp } \xi$. Therefore

$$p y^i > p w^i \quad (4)$$

and

$$p y^i > p \bar{x}^i \quad (5)$$

for every $i \in \text{supp } \xi$. Multiplying (4) and (5) by t_i and q_i respectively, and summing over all $i \in N$ leads to a contradiction with (3). Thus, we only need to prove that

$$C_{fr}(\mathcal{E}) \subseteq W_d(\mathcal{E}).$$

Let $\bar{x} = (\bar{x}^i)_{i \in N} \in C_{fr}(\mathcal{E})$. We want to show that for some prices $p \in \mathbb{R}^L$ and for some vector of dividends $d \in \mathbb{R}_+^N$ the triple (\bar{x}, p, d) satisfies the conditions of attainability and individual rationality. To this end, consider the sets

$$G_w^i = \{y^i - w^i | y^i \in \mathcal{P}_i(\bar{x}^i)\}, \quad i \in N,$$

and

$$G_{\bar{x}}^i = \{y^i - \bar{x}^i | y^i \in \mathcal{P}_i(\bar{x}^i)\}, \quad i \in N.$$

Denote the convex hull of the union of these sets by G :

$$G = \text{conv} \bigcup_{i \in N} (G_w^i \cup G_{\bar{x}}^i).$$

We claim that $0 \notin G$. Suppose otherwise. Then, by convexity of the sets G_w^i and $G_{\bar{x}}^i$ there exist $t = (t_i)_{i \in N}$, $q = (q_i)_{i \in N}$, $y, z \in X$ such that

$$\begin{aligned} \sum_{i \in N} t_i + \sum_{i \in N} q_i &= 1, \quad t_i, q_i \geq 0, \quad i \in N, \\ \sum_{i \in N} t_i y^i + \sum_{i \in N} q_i z^i &= \sum_{i \in N} t_i w^i + \sum_{i \in N} q_i \bar{x}^i, \\ y^i &\in \mathcal{P}_i(\bar{x}^i), \quad i \in \text{supp } t, \quad z^i \in \mathcal{P}_i(\bar{x}^i), \quad i \in \text{supp } q \end{aligned} \quad (6)$$

Notice that the left part of (6) is equal to

$$\sum_{i \in N} t_i y'^i + \sum_{i \in N} q_i y'^i,$$

where

$$y'^i = \frac{t_i}{t_i + q_i} y^i + \frac{q_i}{t_i + q_i} z^i, \quad i \in \text{supp } t \cup \text{supp } q.$$

Since by convexity of preferences $y'^i \in \mathcal{P}_i(\bar{x}^i)$, we have rejection of \bar{x} by the fuzzy coalition $(\xi_i)_{i \in N}$ with

$$\xi_i = t_i + q_i, \quad i \in N,$$

a contradiction. Hence, a zero point does not belong to the convex set G . By the separating hyperplane theorem there exists $p \in \mathbb{R}^L \setminus \{0\}$ such that for every $y^i \in \mathcal{P}_i(\bar{x}^i)$, $i \in N$

$$py^i \geq pw^i, \quad (7)$$

and

$$py^i \geq p\bar{x}^i. \quad (8)$$

Define the components of the vector d by

$$d_i = \max \{0, p\bar{x}^i - pw^i\}, \quad i \in N. \quad (9)$$

Then $\bar{x}^i \in B_i(p, w^i, d_i)$, $i \in N$. To show the individual rationality of \bar{x} suppose that for some $i \in N$ and for some $y^i \in \mathcal{P}_i(\bar{x}^i)$ $py^i = pw^i$. Since w^i belongs to the interior of X_i we can find x^i such that $px^i < pw^i$. Consider $y'^i = \lambda x^i + (1 - \lambda)y^i$. Observe that $py'^i < pw^i$. On the other hand, by continuity of preferences $y'^i \in \mathcal{P}_i(\bar{x}^i)$ provided λ is small enough, which yields a contradiction with (7). Therefore $py^i > pw^i$ for every $y^i \in \mathcal{P}_i(\bar{x}^i)$, $i \in N$. Suppose that $p\bar{x}^i > pw^i$. By using the same argument, $py^i > p\bar{x}^i$ for each $y^i \in \mathcal{P}_i(\bar{x}^i)$, $i \in N$. If not, we can find $y'^i \in \mathcal{P}_i(\bar{x}^i)$ such that $py'^i < p\bar{x}^i$ and obtain a contradiction with (8). So we have that for every agent $i \in N$ and for every consumption bundle $y^i \in X_i$ strictly better than \bar{x}^i

$$py^i > \max\{pw^i, p\bar{x}^i\} = pw^i + d_i.$$

This gives individual rationality and completes the proof of Theorem 1.

□

Note that it is easy to repair the second example of Section 2 so that it will satisfy all the conditions of Theorem 1. Just take for example $X_i = [-10, 10] \times [-10, 10]$, $i = 1, 2, 3$. Nothing will change: a single element of the fuzzy rejective core is $x^1 = (0, 0)$, $x^2 = x^3 = (1, 1)$, which is also the unique competitive equilibrium supported by prices $p = (0, 1)$.

7 References

1. Anderson, R.M. (1992), The core in perfectly competitive economies, in: Handbook of Game Theory, Volume I, Chapter 14, 413–457.
2. Aubin, J.P. (1979), Mathematical methods of game and economic theory, Amsterdam, North-Holland.
3. Aumann, R.J., and J. H. Drèze, (1986), Values of market with satiation or fixed prices, *Econometrica*, 53, 1271–1318.
4. Debreu, G. (1959), Theory of value, Yale University Press, New Haven.
5. Debreu, G., and H. Scarf, (1963), A limit theorem on the core of an economy, *International Economic Review*, 4, 235–246.
6. Drèze, J.H., and H. Müller, (1980), Optimality properties of rationing schemes, *Journal of Economic Theory*, 23, 150–159.
7. Hüsseinov, F. (1994), Interpretation of Aubin’s fuzzy coalitions and their extension, *Journal of Mathematical Economics*, 23, 499–516.
8. Kajii, A. (1996), How to discard non-satiation and free-disposal with paper money, *Journal of Mathematical Economics*, 25, 75–84.
9. Klaus, B., H. Peters, and T. Storcken, (1998), Strategy-proof division with single-peaked preferences and individual endowments, *Social Choice and Welfare*, 15(2), 297–311.
10. Konovalov, A.V., and V. M. Marakulin, (1997), On the finiteness of the set of economic equilibria with non-standard prices, Discussion paper No. 39, Sobolev Institute of Mathematics, SB RAS, Novosibirsk.
11. Makarov, V.L. (1981), Some results on general assumptions about the existence of economic equilibrium, *Journal of Mathematical Economics*, 8, 87–100.
12. Marakulin, V.M. (1988), Equilibrium with nonstandard prices and its properties in mathematical economic models, Discussion paper No. 18, Institute of Mathematics, Siberian Branch of the USSR Academy of Sciences, Novosibirsk.

13. Marakulin, V. (1990), Equilibrium with nonstandard prices in exchange economies, in: R. Quandt, D. Triska, eds., *Optimal decisions in market and planned economies*, Westview Press, London.
14. Mas-Colell, A. (1992), Equilibrium theory with possibly satiated preferences, in: M. Majumdar, ed., *Equilibrium and dynamics: Essays in honor of David Gale*, Macmillan, London.
15. Nielsen, L. T., (1994), Pareto optima in incomplete financial markets, *Journal of Mathematical Economics*, 23, 87–100.
16. Polemarchakis, H., and P. Siconolfi (1993), Competitive equilibria without free disposal or non-satiation, *Journal of Mathematical Economics*, 22, 85–99.
17. Thomson, W., and L. Zhou (1993), Consistent solutions in atomless economies, *Econometrica*, 61, 575–587.